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MATHEMATICAL ANALYSIS OF THE INTENSITY DISTRIBUTION OF OPTICAL IMAGES FOR VARIOUS DEGREES OF COHERENCE OF ILLUMINATION

(Representation of Intensity by Hermitian Matrices)

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MATHEMATICAL ANALYSIS OF THE INTENSITY DISTRIBUTION OF OPTICAL IMAGES FOR VARIOUS DEGREES OF COHERENCE OF ILLUMINATION

(Representation of Intensity by Hermitian Matrices)

Hideya Gamo[(1)]

ABSTRACT. Since optical systems have distinctive features as compared to electrical communication systems, some formulation should be prepared for the optical image in order to use it in information theory of optical systems. In this paper the following formula for the intensity distribution of the image by an optical system having a given aperture constant α in the absence of both aberration and focusing defects is obtained by considering the nature of illumination, namely, coherence, partial coherence and incoherence; /431*

$$I(y)=\sum_n\sum_m a_{nm}\, u_n(y)\, u_m^*(y)$$

$$u_n(y)=\sin\frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right)\Big/\frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right)$$

$$a_{nm}=\left(\frac{2\alpha}{\lambda}\right)^2\iint\Gamma_{12}(x_1-x_2)E(x_1)E^*(x_2)|A(x_1)||A(x_2)|u_n(x_1)u_m(x_2)dx_1dx_2$$

where I(y) is the intensity of the image at a coordinate point y, Γ_{12} the phase coherence factor introduced by H. H. Hopkins etc., E(x) the complex transmission coefficient of the object and A(x) the complex amplitude of the incident waves at the object, and the integration is taken over the object plane.

The above expression has some interesting features; namely, the "intensity matrix" composed of the element a_{nm} mentioned above is a positive-definite Hermitian matrix, and the diagonal elements are given by the intensities sampled at every point of the image plane separated by the distance $\lambda/2\alpha$, and the trace of the matrix or the sum of diagonal elements is equal to the total intensity integrated over the image plane. Since

*Numbers in the margin indicate pagination in the original foreign text.

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an Hermitian matrix can be reduced to diagonal form by a unitary transformation, the intensity distribution of the image can be expressed as

$$I(y) = \lambda_1 |\sum S_{i1} u_i|^2 + \lambda_2 |\sum S_{i2} u_i|^2 + \cdots\cdots + \lambda_n |\sum S_{in} u_i|^2 + \cdots\cdots$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are non-negative eigenvalues of the intensity matrix. In case of coherent illumination, only the first term of the above equation remains and all the other terms are zero, because the rank of the coherent intensity matrix is one, and its only non-vanishing eigenvalue is equal to the total intensity of the image. On the other hand, the rank of the incoherent intensity matrix is larger than the rank of any other coherent or partially coherent cases. The term of the largest eigenvalue in the above formulation may be especially important, because it will correspond /432
to the coherent part of the image in case of partially coherent illumination.

From the intensity matrix of the image obtained by uniform illumination of the object having uniform transmission coefficient, we may derive an interesting quantity, namely

$$d = -\sum_n (\lambda_n/I_0) \log (\lambda_n/I_0)$$

where λ_n is the n^{th} eigenvalue of the intensity matrix and I_0 is the trace of the matrix. d is zero for the coherent illumination and becomes logN for the incoherent illumination, where N is the "degree of freedom" of the image of the area S, namely, $N = 4\alpha^2 S/\lambda^2$. The value of d for partially coherent illumination is a positive quantity smaller than log N. A quantity $\delta = (d_0 - d)/d_0$ may be regarded as a measure of the "degree of coherence" of the illumination, where $d_0 = \log N$ and δ is unity for the coherent case and zero for perfectly incoherent case.

The sampling theorem for the intensity distribution is derived, and the relation between elements of intensity matrix and intensities sampled at every point separated by the distance $\lambda/4\alpha$ is given.

1. INTRODUCTION

The relationship between the image and the object in optics in terms of information theory is a recent topic. Even with an established foundation, such as treatment by response functions, we are still on the threshold of dealing with such traditional problems in information theory as entropy or noise. While we have rather advanced knowledge of electric communications, much of it cannot be readily applied to optics. There is a need to consider specific optical characteristics and make new formulations.

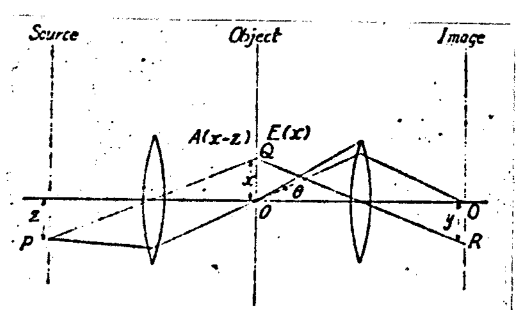
From this point of view, optics is regarded as having the following characteristics: (i) the directly observable quantity is not wave amplitude, but the square of its absolute value, intensity. Intensity is never negative. (ii) The amplitude and phase of the response function in optics regarded as a space filter are both independent physical quantities. In electrical circuits, temporal variations occur and there are restrictions due to cause and effect. (iii) The information quantity, including the image, varies with the illumination. For example, with coherent illumination we can extract information on wave amplitude and phase, while with incoherent illumination, we cannot extract phase information. (iv) In determining the capacity of optics as a path of communications, noise is a basic physical quantity including the effects of stray light, "seeing" through irregular variations in the medium, granularity of the film, the physiological process of sight, etc. In all of these processes, although we either add a new intensity to an existing one or subtract from it, the overall intensity can never become "negative". (v) It is a multidimensional, especially a two-dimensional, space filter.

These facts are important starting points in building an information theory for optics and a discussion which ignores them may serve as a guide line, but it can never be conclusive.

The author considered the first and second characteristics given above for a one-dimensional, aberrationless system and studied the changes in the informational properties of the image depending on the third problem, the degree of coherence, through the use of the sampling theorem known in communications theory. The intensity distribution of the image is described in terms of an intensity matrix, a positive Hermitian matrix, from which were derived various physical concepts. H. H. Hopkins' "phase coherence factor" is also used.

2. THE SAMPLING THEOREM

As a basis for later treatment, let us give a simple explanation of the sampling theorem. First, let us consider the optical system formed of the light source, object, lens and image. The wave from the source with unit intensity at point P above the source and incident at point Q on the object plane is designated by A(x–z). If the complex transmittivity of the object is E(x), the amplitude of the wave after transmission is E(x)A(x–z). If we denote one half of the aperture angle of the light bundle as θ, and the wavelength by λ, /433
then 2 sin ρ/λ will correspond exactly to the band width of the filter in an electric communications system. If we find the Fourier transformation value f(X) of the complex amplitude of the above wave, we have:



$F(y)$: complex amplitude of image
$A(x-z)$: complex amplitude of the incident wave from a point source at P
$E(x)$: complex transmission coefficient

Figure 1

$$f(X,z) = \int_{-\infty}^{+\infty} E(x)A(x-z)e^{-2\pi iXx}dx \qquad (1)$$

Here, since X = sin θ/λ, corresponding to the direction cosine showing the direction of progress of the plane wave, f(X) is none other than the amplitude of the plane wave progressing in the direction of sin θ = λX. However, f(X), which contributes to the image depending on the size of the aperture in the optical system, is restricted to a certain zone width. Or, from 2 sin θ/λ above,

$$-\alpha/\lambda \leqq X \leqq \alpha/\lambda \tag{2}$$

Here, $\alpha = \sin\theta$. Thus, since the domain of $f(X)$ is confined to $2\alpha/\lambda$, we can use a Fourier series expression:

$$f(X,z) = \sum_{n=-\infty}^{+\infty} a_n(z) e^{-2\pi i n X} \tag{3}$$

$$a_n(z) = \frac{\lambda}{2\alpha} \int_{-\frac{\alpha}{\lambda}}^{+\frac{\alpha}{\lambda}} f(X,z) e^{+2\pi i \cdot \frac{n\lambda}{2\alpha}} dx \tag{4'}$$

As to $F(y)$, the amplitude distribution in the image plane, one needs to find the $f(X)$ Fourier inverse series with domain $2\alpha/\lambda$, and using Equation (3):

$$F(y,z) = \int_{-\frac{\alpha}{\lambda}}^{+\frac{\alpha}{\lambda}} f(X,z) e^{2\pi i X y}\, dX \tag{4}$$

$$= \sum_{n=-\infty}^{+\infty} a_n(z) \frac{\sin \frac{2\pi\alpha}{\lambda}\left(y - \frac{n\lambda}{2\alpha}\right)}{\left(y - \frac{n\lambda}{2\alpha}\right)} \tag{4'}$$

However, as seen from Equation (4), the coefficient $a_n(z)$ is none other than the image amplitude $F(n\lambda/2\alpha)$ in $n\lambda/2\alpha$ multiplied by $\lambda/2\alpha$. Thus,

$$F(y) = \sum_{n=-\infty}^{+\infty} F\left(\frac{n\lambda}{2\alpha}, z\right) \frac{\sin \frac{2\pi\alpha}{\lambda}\left(y - \frac{n\lambda}{2\alpha}\right)}{\frac{2\pi\alpha}{\lambda}\left(y - \frac{n\lambda}{2\alpha}\right)} \tag{5}$$

The complex amplitude of the image obtained through an optical system with a zone width $2\alpha/\lambda$ determined by the aperture is readily determined from the complex amplitudes at points $\lambda/2\alpha$ apart. Thus the system of equations $\{u_n = \sin \frac{2\pi\alpha}{\lambda}(y - n\lambda/2\alpha) / \frac{2\pi\alpha}{\lambda}(y - n/2\alpha)\}$ is a completely orthogonal system. Or, for any arbitrary positive or negative integers n, m,

$$\int^{+\infty} u_n u_m \, dy = \frac{\lambda}{2\alpha} \delta_{nm} \quad (n=m,\ 1;\ n \neq m,\ 0) \tag{5'}$$ [2]

The results are exactly the same as the sampling theorem for electrical signal waves with limited frequency ranges, differing only at points where the amplitudes being sampled are complex numbers. Such an appearance of complex amplitudes in the optical system is related to the second characteristic cited in § 1. The results for the one-dimensional, aberrationless system thus obtained can easily be expanded to cover a two-dimensional, aberrationless system and if we use

$$して\left\{\sin 2\pi\frac{\alpha}{\lambda}(\xi - n\lambda/2\alpha) \Big/ \frac{2\pi\alpha}{\lambda}(\xi - \lambda n/2\alpha)\right\}\left\{\sin \frac{2\pi\alpha}{\lambda} (\eta - m\lambda/2\alpha) \Big/ \frac{2\pi\alpha}{\lambda}(\eta - n\lambda/2\alpha)\right\}$$

as the standard function, then the complex amplitude limited to any given range can be shown in terms of an expanded series. Rigorous treatment in the case of a round aperture is painstaking but it can be accomplished by sampling of the points on the lattice described above. For example, if we take the lattice points determined by a rectangular aperture circumscribing the given circle, the amplitude distribution is determined by all the sampling values but the values of each sampling can never be independent.

The discussion given above covers the case of illumination by a point source and holds true only for such coherent illumination. And even in the case of coherent illumination, it has a clear physical meaning only in the case of independent derivation of amplitude and phase by a suitable phase differential method. This is due to the fact that only intensity can be observed directly, in accordance with the first property in § 1. Below we shall discuss a method of finding physical properties from the intensity distribution of the image in cases where the illumination is either partially coherent or totally incoherent.

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[2] See Appendix 3.

3. INTENSITY MATRIX AND PHASE COHERENCE FACTOR

The intensity distribution in the case of coherent illumination is derived by Equation (5) in § 2. If we denote the intensity at a point on the image plane R (coordinate y) by I(y), then

$$\begin{aligned} I(y) &= F(y)\,F^*(y) \\ &= \sum_n \sum_m F\left(\frac{n\lambda}{2\alpha}, z\right) F^*\left(\frac{m\lambda}{2\alpha}, z\right) u_n(y)\, u_m^*(y) \end{aligned} \tag{6}$$

Here,

$$u_n(y) = \sin \frac{2\pi\alpha}{\lambda}\left(y - \frac{n\lambda}{2\alpha}\right) \Big/ \frac{2\pi\alpha}{\lambda}\left(y - \frac{n\lambda}{2\alpha}\right)$$

In the following, the intensity distribution of the image is considered for the general case where the source is of finite size and brightness at each point may take any arbitrary value. If we designate the brightness at a point P on the source (coordinate z) by J(z), the intensity at a point R (coordinate y) on the image plane is given by:

$$I(y) = \sum \sum a_{nm}\, u_n(y)\, u_m(y) \tag{7}$$

$$a_{nm} = \int J(z)\, F\left(\frac{n\lambda}{2\alpha}, z\right) F^*\left(\frac{m\lambda}{2\alpha}, z\right) dz \tag{7'}$$

For, since the light rays from each point of the source are mutually inco- /434
herent, the sum of the intensities at each point source is the intensity of the whole. This fact can be understood from a statistical point of view in the following manner. Since intensity is the squared time average value of the light wave, there is no correlation between the light waves from the light rays at each point and the squared average values for the overlapping waves are each equal to the sum of the squared average values.

The results for Formula (7) above constitute the major portion of this paper. Formula (7) is a quadratic equation with variables u_n and u_m and coefficient a_{nm}. And, since intensity cannot be negative, this is a positive quadratic equation. The matrix formed by the coefficient a_{nm} is referred to here as the intensity matrix[(3)]. Given this matrix, since the intensity distribution for the image is defined, we can say that the intensity matrix includes all information on the intensity distribution of the image. Below, we consider the physical properties of these intensity matrices.

Let us consider the expression of the intensity matrix element a_{nm} in terms of H. H. Hopkins' phase coherence factor. The latter has been thoroughly studied already and is of considerable convenience in treatment. First, let us transform a_{nm} as defined by (7) as follows:

$$a_{nm}=\iiint J(z)E(x_1)E^*(x_2)A(x_1-z)A^*(x_2-z)\, u\left(x_1-\frac{n\lambda}{2\alpha}\right)u^*\left(x_2-\frac{m\lambda}{2\alpha}\right)dx_1dx_2dz \tag{8}$$

Here, $u(x_1 - n\lambda/2\alpha)$, $u(x_2 - mx/2\alpha)$ is the amplitude of the image produced at point R, $x = n\lambda/2\alpha$, $m\lambda/2\alpha$ of the image plane when a wave with unit amplitude at points Q_1, Q_2 in the object plane passes through an optical system of zone width $2\alpha/\lambda$ (α: numerical aperture, λ: wavelength). In the aberrationless system considered here,

$$u\left(x_1-\frac{n\lambda}{2\alpha}\right)=\sin\frac{2\pi\alpha}{\lambda}\left(x_1-\frac{n\lambda}{2\alpha}\right)\Big/\pi\left(x_1-\frac{n\lambda}{2\alpha}\right) \tag{8'}$$

(For proof, see appendix 1).

In the form of (8), the relationship with the phase coherence factor [2] becomes clear. If we take the integral $I(x_1,x_2)$ for the light source position Z included in Equation (8),

$$I(x_1,x_2)=\int J(z)A(x_1-z)A^*(x_2-z)dz \tag{9}$$

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(3) The intensity matrix has features in common with the information matrix proposed by D. M. MacKay [7] with no specific object. N. Wiener proposed a coherence matrix, treating the lightwave as a time series [8], but it has not been applied in optical systems.

This quantity is a kind of correlative function for the complex amplitude at x_1, x_2 and if we let $U_{1r}U_2$ indicate the amplitude at each point due to source Z, then it can be expressed as:

$$I(x_1, x_2) = \int U_1 U_2^* \, dz \tag{9'}$$

If I_1, I_2 are the intensities at points x_2, x_2,

$$I(x_1, x_2) = \sqrt{I_1 I_2}\,\Gamma(x_1 - x_2) \tag{9''}$$

Here $\Gamma(x_1 - x_2)$ is the phase coherence factor, which, by definition, is:

$$\Gamma(x_1 - x_2) = \frac{1}{\sqrt{I_1 I_2}} \int U_1 U_2^* dz \tag{10}$$

Since I_1, I_2 are none other than $|A(x_1)|^2$, $|A(x_2)|^2$, the squares of the absolute values of the complex amplitude of the incident waves at points x_1, x_2 of the object plane due to light source

$$\begin{aligned} a_{nm} = \iint & \Gamma(x_1 - x_2) E(x_1) E(x_2) |A(x_1)||A(x_2)| \\ & u\left(x_1 - \frac{n\lambda}{2\alpha}\right) u^*\left(x_2 - \frac{m\lambda}{2\alpha}\right) dx_1\, dx_2 \end{aligned} \tag{11}$$

Here, $E(x)$ is the complex transmissivity, $A(x)$ is the complex amplitude of the incident wave, $u\left(x_1 - \frac{n\lambda}{2\alpha}\right)$ is the amplitude of the wave produced at point $n\lambda/2\alpha$ of the image plane by the wave with unit amplitude at point x_1 of the object plane (Equation 8'), and $\Gamma(x_1 - x_2)$ is the phase coherence factor. According to Hopkins, Γ is given by

$$\begin{aligned} \Gamma(X_1 - X_2, Y_1 - Y_2) = & \frac{1}{\sqrt{I_1 I_2}} \\ & \iint J(x, y) e^{i[x(X_1 - X_2) + y(Y_1 - Y_2)]} dx\, dy \end{aligned} \tag{12}$$

$J(x,y)$ is the brightness at points (x,y) on the light source. We know that illumination using a condenser lens can also be expressed by an equation like that given by a suitable equivalent light source.

Equation (11) above is the first of our desired results. Next, let us consider several actual cases of intensity matrices $||\alpha_{nm}||$.

4. EXAMPLES OF INTENSITY MATRICES

4.1. Coherent Illumination

The phase coherence factor $\Gamma(x_1 - x_2)$ in this case is always 1, regardless of x_1, x_2 [2]. Consequently, if we set $\Gamma = 1$ in Equation (12), we find that this integral is separated into the product of two integrals, or

$$a_{nm} = F\left(\frac{n\lambda}{2\alpha}\right)F^*\left(\frac{m\lambda}{2\alpha}\right)$$

The products of the complex amplitudes at points $n\lambda/2\alpha$, $m\lambda/2\alpha$ in the image plane are none other than the matrix elements. This brings us back to Equation (6) in § 3 and it is important in the discussion below to consider the meaning of the matrix in terms of these elements. A conspicuous feature of the intensity matrix in this case is that the matrix rank is one, and the eigenvalue which is not zero is (Σa_{nm}). When the minor determinant formed by the matrix elements is zero for some degree lower than r and the minor determinants of degrees greater than (r + 1) are always zero, the rank of this matrix is said to be r. It is easily seen that the second or higher degree minor determinants with elements given by Equation (13) will all be zero. And, therefore, the formula for finding the eigenvalues (characteristic equation) /435

$$\det|a_{nm} - \lambda\delta_{nm}| = 0$$

becomes $x^N - (\Sigma a_{nm})x^{N-1} = 0$ and only one of the eigenvalues will not be zero. This is equal to the trace (spur) of the matrix.

4.2. Incoherent Illumination

The phase coherence factor Γ in this case is:

$$\Gamma(x_1-x_2)=\begin{cases}0 & x_1\neq x_2\\ 1 & x_1=x_2\end{cases}$$

If these conditions are used in Equation (11), the element a_{nm} of the intensity function in the case of incoherent illumination can be expressed as a single integral, as follows:

$$a_{nm}=\int_{-\infty}^{+\infty} I(x)\,u\left(x-\frac{n\lambda}{2\alpha}\right)u^{*}\left(x-\frac{m\lambda}{2\alpha}\right)dx \tag{14}$$

Here,

$$u\left(x-\frac{n\lambda}{2\alpha}\right)=\sin\frac{2\pi\alpha}{\lambda}\left(x-\frac{n\lambda}{2\alpha}\right)\Big/\pi\left(x-\frac{n\lambda}{2\alpha}\right)$$

The computation of this equation is closely related to the well-known integral

$$I(y)=\int_{-\infty}^{+\infty} I(x)\,|u(x-y)|^2\,dx \tag{14'}$$

which gives the image intensity I(y) with an incoherent source [brightness I(x)]. According to the latter, one integral would be enough, while according to the intensity matrix the number of necessary integrals increases, leading us to suspect needless complexity. We shall consider this point at the conclusion.

The simplest case is that where the brightness distribution at the object is uniform: if I(x) = A (constant),

$$\begin{aligned}a_{nm}&=A\int_{-\infty}^{+\infty} u\left(x-\frac{n\lambda}{2\alpha}\right)u^{*}\left(x-\frac{m\lambda}{2\alpha}\right)dx\\&=\frac{2\alpha}{\lambda}A\,\delta_{nm}\ (\delta_{nn}=1,\ n=m;\ \delta_{nm}=0,\\&\quad n\neq m)\end{aligned} \tag{15}$$ [4]

(4) Determined by I(y) from Equation (14') and (15). Should be consistent with I(y). See Appendix 2).

Another simple and fundamental case is that where the amplitude distribution at the object plane varies sinusoidally in space. The brightness at the object plane is

$$I(x)=A\cos^2\omega x=\frac{1}{2}A(1+\cos 2\omega x) \tag{15}$$

In this case the intensity matrix element a_{nm} is the integral

$$a_{nm}=\frac{1}{2}A\int_{-\infty}^{+\infty}(1+\cos 2\omega x)\times\frac{\sin\frac{2\pi\alpha}{\lambda}\left(x-\frac{n\lambda}{2\alpha}\right)\sin\frac{2\pi\alpha}{\lambda}\left(x-\frac{m\lambda}{2\alpha}\right)}{\pi\left(x-\frac{n\lambda}{2\alpha}\right)\pi\left(x-\frac{m\lambda}{2\alpha}\right)}dx \tag{16}$$

Transforming the independent variable to $2\pi\alpha x/\lambda = \xi$ and letting $\lambda\omega/2\pi\alpha = p$, we obtain:

$$a_{nm}=A\frac{\alpha}{\pi\lambda}\int_{-\infty}^{+\infty}(1+\cos 2p\xi)\frac{\sin(\xi-n\pi)}{\xi-n\pi}\frac{\sin(\xi-n\pi)}{\xi-m\pi}d\xi.$$

Since this type of definite integral occurs frequently in the examples given below and many of them are not found in common tables of integrals, the method of integration and the chief results are compiled in Appendix 3. Let us briefly show the results. When $0 \geqq p \geqq 1$, or when $(0 < \omega < 2\pi\alpha/\lambda)$, $n \neq m$

$$a_{nm}=A\frac{\alpha}{\lambda}\frac{\sin(n-m)\pi(1-p)}{(n-m)\pi}\cos(n+m)p\pi$$
$$a_{nn}=\frac{A}{2}[1+(1-p)\cos 2np\pi]\quad n\equiv m \tag{16'}$$

when $p > 1$, or $\omega > 2\pi\alpha/\lambda$, then $a_{nm} = 0$. The results which become zero above a certain frequency are consistent with the fact that, when the image intensity distribution Equation (14') is treated as a response function, a triangular response function occurs [1] and the values higher than the frequency range $4\pi\alpha/\lambda$ become zero.

Let us consider the rank and eigenvalues of intensity matrices on the basis of these results. When object brightness is uniform, the intensity matrix for Equation (15) will clearly be a diagonal matrix, the diagonal elements are not equal to zero. Of the cases where brightness varies periodically, if we take the one where $p = 1/2$, or $\omega = \alpha/\lambda$, the matrix will likewise be diagonal, with diagonal elements equal to $1/2A(\lambda/2\alpha)\ (1 + 1/2\ (-1)^n)$. As seen from these examples, the rank of the intensity matrix in coherent illumination will be 1, while the value will be larger in the case of incoherent illumination[5]. /436

4.3. Partially Coherent Illumination

In the one-dimensional problem we have just treated, the phase coherence factor is given by

$$\Gamma(x_1-x_2)=\frac{\sin\frac{2\pi\alpha}{\lambda}S(x_1-x_2)}{\frac{2\pi\alpha}{\lambda}S(x_1-x_2)} \tag{17}$$

in accordance with Equation (12). S is a parameter showing the size of the light source. Naturally at the limit $S \to 0$, $\Gamma = 1$. If we let α in Equation (17) remain equal to the numerical aperture in the system considered so far, $S = 1$ may be thought to mean that the size of the aperture when the source is seen from the object is equal to the numerical aperture in the system in question. From the results given below, one sees that the conditions where $S > 1$ or $S < 1$ are very important.

Let us find the elements of the intensity matrix for an object image with ordinary transmittivity. Substituting $I(x_1 - x_2)$ given by (17) in (11), and letting $E(x)\ A(x) = K$,

$$a_{nm}=\frac{K^2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}\frac{\sin S(\xi-\eta)}{S(\xi-\eta)}\,\frac{\sin(\xi-n\pi)}{\xi-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\xi d\eta \tag{18}$$

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[5] When the size of the object in question is finite, variation in the intensity distribution or transmissivity of the object plane can be expressed as a Fourier series and the results of § 4.2 and § 4.3 can be applied to each term.

Here, $\xi = 2\pi\alpha x_1/\lambda$, $\eta = 2\pi\alpha x_2/\lambda$. The results of this integral are as follows (see Appendix 3):

$$S>1 \qquad a_{nm} = \frac{K^2}{S}\,\delta_{nm} \tag{18'}$$

$$S<1 \qquad a_{nm} = K^2\,\frac{\sin S(n-m)\pi}{S(n-m)\pi} \tag{18''}$$

Next, let us consider the case where the wave amplitude in the object plane varies sinusoidally. Substituting $\Gamma(x_1 - x_2)$ as given by Equation (17) in Equation (11) and letting $E(x)\ A(x) = K \cos \omega x$,

$$a_{nm} = \frac{K^2}{\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos p\xi \cos p\eta \; \frac{\sin S(\xi-\eta)\sin(\xi-n\pi)\sin(\eta-m\pi)}{S(\xi-\eta)(\xi-n\pi)(\eta-m\pi)}\,d\xi\,d\eta \tag{19}$$

Here, $\xi = 2\pi\alpha x_1/\lambda$, $\eta = 2\pi\alpha x_2/\lambda$, $p = \lambda\omega/2\pi\alpha$. The details of computation are given in the appendix and the results are shown here.

(i) $p>(1+S)$, ($S>1$ または $S>1$)

$$a_{nm} \equiv 0$$

(ii) $(1+S)<p<1$ ($S>1$, 又は $S>1$) (20-1)

$$\left.\begin{aligned} a_{nm} &= \frac{K^2}{2S}\,\frac{A}{(n-m)\pi} \qquad (n \neq m) \\ a_{nm} &= \frac{K^2}{4S}(1+S-p) \end{aligned}\right\} \tag{20-2}$$

(depending on whether $S > 1$ or $S < 1$)

$$\begin{aligned} a_{nm} &= \frac{K^2}{2S}\,\frac{A+B}{(n-m)\pi} \qquad (n \neq m) \\ a_{nm} &= \frac{K^2}{4S}\left[1+S-p+2(1-p)\cos 2np\pi\right] \end{aligned} \tag{20-3}$$

Here, A and B are:

$$\begin{aligned} A &= \cos np\pi \sin (n-m)(S+1-p)\frac{\pi}{2} \\ &\quad \cos\left[(n+m)p+(S-1)(n-m)\right]\frac{\pi}{2} \\ &\quad -\sin np\pi \sin (n-m)(p+1-S)\frac{\pi}{2} \\ &\quad \sin\left[(n+m)p+(S-1)(n-m)\right]\frac{\pi}{2} \end{aligned} \tag{20-4}$$

(Equation continued on next page)

$$2B = \sin(n-m)(1-p)\pi \cos(n+m)p\pi - \sin(n-m)p\pi \cos[(n+m)p+n-m]\pi$$

$$\text{(iii)} \quad S>1, \quad (S-1)>p>1 \tag{20-4}$$

$$a_{nm} = 0 \quad (n \neq m).$$

$$a_{nn} = \frac{K^2}{4S}$$

$$1>p>0$$

$$a_{nm} = \frac{K^2}{4S} \frac{\sin(n-m)(1-p)\pi}{(n-m)\pi} \cos(n+m)p\pi \quad (n \neq m) \tag{20-5}$$

$$a_{nn} = \frac{K^2}{4S}[1+(1-p)\cos 2np\pi]$$

$$\text{(iv)} \quad S<1, \quad (1-S)>p>0$$

$$a_{nm} = K^2 \cos^2 p\eta\pi \frac{\sin S(n-m)\pi}{S(n-m)\pi} \quad (n \neq m),$$

$$a_{nn} = K^2 \cos^2 p\eta\pi \qquad (20\text{-}5)$$

We notice several interesting points when looking at the case of partially coherent illumination as described above. First, with incoherent illumination, when $p > 1$, $\omega > 2\pi\alpha/\lambda$, $a_{nm} = 0$ and no intensity variations occur but with semicoherent illumination, they are difficult to distinguish since $p > (1 - S)$, $\omega > (1 + S)2\pi\alpha/\lambda$, and they are effective over a greater range than in the preceding case. Secondly, when $S < 1$, $(S - 1) > p > 1$ we obtain results in exactly the same form as incoherent illumination in (iii). Thirdly, when $S < 1$, $(1 - S) > p > 0$, the results of (iv) are found to revert to the case of coherent illumination at the limit $S \to 0$. The determination of the matrix rank according to the above matrix elements and the eigenvalues is very difficult, except in special cases. This fact limits the practical significance of treatment through intensity matrices. However, since the intensity matrices themselves have the interesting, general properties described below, /437 they are useful in organizing and clarifying the physical concepts concerning optical images.

Above, we dealt only with cases where the transmissivity or brightness varied periodically but for images of objects with arbitrarily varying transmittivities or brightnesses, the transmissivity or brightness can be expressed as integrals of periodic components by Fourier integrals and handled through calculations closely paralleling those cited. Thus is is not difficult to formulate intensity matrix elements for objects possessing general transmissivity distributions.

5. GENERAL PROPERTIES OF INTENSITY MATRICES

The intensity matrix was introduced through the process of finding the square of the amplitude, using the sampling theorem for transmission systems with limited frequency ranges. In electric communications, the square of the amplitude is a quantity proportional to electric power, and a formula coinciding with the intensity matrix emerges, but this problem need not concern us directly here. This is because the coherence of illumination in an optical system is a question specific to the optical system, as described in § 1.

In order to simplify comprehension of the discussion which follows, let us write Equation (7) once more. The intensity I(y) is

$$I(y)=\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} a_{nm}\, u_n(y)\, u_m(y)$$
$$u_n(y)=\sin \frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right)\Big/\frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right) \tag{7}$$

$u_1, u_2, \ldots, u_n, \ldots$ constitutes one vector of a multidimensional space, the total of $\Sigma a_{nm} u_m = v_n$ is another vector, and the intensity is the scalar product of the two vectors.

$$I(y)=(\phi,\, A\phi) \tag{21}$$

ϕ is the vector $\{u_1, u_2, \ldots, u_n, \ldots\}$ and A is the matrix with elements a_{nm}.

For the dimension number (N) of the above multidimensional space related to the image with which we are concerned, we may consider the "degree of freedom" of the image. The degree of freedom N is none other than the number of sampling points expressing the image in question. If S is the area of the image, then $N = 4\alpha^2 S/\lambda^2$. This number N corresponds to the degree of freedom defined by Toraldo di Francia in the case of coherent illumination. However, Toraldo adopts another definition in the case of incoherent illumination. In comparison, our method of defining image degree of freedom by the dimension number determining intensity is more consistent mathematically. All the problems of illumination coherence are then included in the properties of the intensity matrices. This enables us to separate the dimension number of the space determined by the numerical aperture in the optical system from the problem of illumination coherence.

Let us first discuss the main points of the physical properties expressed through intensity matrices. The first is that the intensity we are observing is a real number and it is never negative. From this basic requirement, the mathematical properties of intensity matrices are induced.

(i) Intensity matrices are positive Hermitian matrices.

Hermitian matrices are those whose members satisfy the relationship

$$a_{nm} = a^*_{mn} \tag{22}$$

which is easily deduced, setting I(y) and I*(y) equal in Equation (7). The fact that they are positive matrices is due to the fact that y in Equation (7) is never negative [6]. Using the mathematical results regarding positive Hermitian matrices known to us through these properties, we can carry out further studies using intensity matrices.

Next let us consider the direct relationships between the elements of intensity matrices and the observable physical quantities. One of these is:

(ii) The diagonal elements of an intensity matrix are equal to the sampling values of intensity at $\lambda/2\alpha$ intervals, and the trace of the intensity matrix (sum of diagonal elements) is equal to the integral intensity of the image.

---

[6] The condition therefore is that the primary determinants of matrix $||a_{nm}||$ are all positive.

The first property is found from the fact that function $u_n(y)$ included in Equation (7) is one at sampling point $y = n\lambda/2\alpha$, and zero at other sampling points. The quantity called the integral intensity is the image intensity integrated over the entire image plane. Using Equation (5'), the integral in (7) becomes

$$I_0 = \int I(y)\,dy = \frac{\lambda}{2\alpha}\left(\sum_{n=-\infty}^{+\infty} a_{nn}\right) \tag{23}$$

Illumination coherence is included in the properties of the intensity matrix, but it is specifically represented through diagonalization of the matrix by a unitary transformation. In representing the intensity I as the scalar product of vectors as in Equation (21), the vectors with basic coordinates for the representation of (12) are denoted by (13). However, it is a rectangular vector with components like ϕ_1 (1, 0, 0, 0, ...), ϕ_2 (0, 1, 0, 0, ...), etc. The vector ϕ given by these basic vectors can be expressed as

$$\phi = x_1\phi_1 + \cdots\cdots + x_\nu\phi_\nu + \cdots\cdots \tag{24}$$

The problem here is to use a rectangular transformation of basic coordinates to change Equation (21) to the simplest form. The new basic vector satisfying such demands is called the eigenvector of the given intensity matrix. In /438
other words, the eigenvectors ψ_1, ψ_2, ..., ψ_i, satisfy the following relationship

$$A\psi_i = \lambda_i\psi_i \tag{25}$$

λ_i is a constant, called the eigenvalue, and it is easily proved that the eigenvalue of a Hermitian matrix is a real number. If we call the components of eigenvector ψ (S_1, S_2, S_3, ...), the formula for the eigenvalue and the eigenvector $A\psi = \lambda\psi$ can be written as follows,

$$\left.\begin{array}{l} (a_{11}-\lambda)S_1 + a_{12}S_2 + a_{13}S_3 + \cdots\cdots + a_{1n}S_n = 0 \\ a_{21}S_1 + (a_{22}-\lambda)S_2 + a_{23}S_3 + \cdots\cdots + a_{2n}S_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}S_1 + a_{n2}S_2 + \cdots\cdots \qquad + (a_{nn}-\lambda)S_n = 0 \end{array}\right\} \tag{26}$$

Since this system of equations for S_1, S_2, ... has solutions other than zero, the determinants made up of the coefficients must be zero. The formula is called the eigenformula and λ_1, λ_2, ... are to be found as its solution. If we find the solution of the system equations with the respective eigenvalues substituted in Equation (26), we obtain the eigenvector belonging to those eigenvalues. Letting (S_{1i}, S_{2i}, S_{3i}, ..., S_{ni}) be the components of eigenvector ψ_i of eigenvalues λ_i, the square of the absolute value of this vector is given by (ψ_i, ψ_i) and the vector, normalized to equal 1, is adopted. And it can be proved that the eigenvectors are mutually orthogonal [5]. Thus the norms and orthogonal conditions can be written

$$(\psi_i, \psi_j) = S_{1i}^* S_{1j} + S_{2i}^* S_{2j} + \cdots\cdots + S_{ni}^* S_{nj} = \delta_{ij} \tag{27}$$

Returning to the beginning, by transforming the basic coordinate vector from ϕ_1, ϕ_2, ..., ϕ_n to the above eigenvector ψ_1, ψ_2, ..., ψ_n the vector $\phi = (u_1, u_2, \ldots, u_n)$ can be expressed by the new vector $\psi = (\xi_1, \xi_2, \ldots, \xi_n)$, or:

$$\phi = \xi_1 \psi_1 + \xi_2 \psi_2 + \cdots\cdots + \xi_n \psi_n \tag{28}$$

Writing the vector components:

$$u_i = \sum_k S_{ik} \xi_k \tag{29}$$

and thus, if the matrix with component S_{ik} in row i and column k is shown by S, we may write

$$\phi = S\psi \tag{30}$$

The coefficients in Equation (28), or the components for the new coordinate system of vector ϕ can be written, by virtue of the rectangular nature of eigenvector ψ_i (27)

$$\xi_i = (\psi_i, \phi) \tag{31}$$

and using the vector components ψ_i, ϕ, in the form of

$$\xi_i = \sum S_{ki}^* x_k \tag{31'}$$

S_{ki}^* in Equation (31') is the component in row i, column k of the inverse transformation matrix, but this matrix has conjugate complex numbers for each element, with rows and columns transposed in matrix S for Equation (29). Generally, the transposition of rows and columns in a matrix is indicated by dashes, but we are expressing the transformed matrix for Equation (31) by S'*. Thus,

$$\phi = S'^* \phi \tag{32}$$

If we consider the square of the absolute value of vector ϕ,

$$(\phi, \phi) = \sum |x_k|^2 \tag{28}$$

However, from Equation (28),

$$(\phi, \phi) = \left(\sum \xi_k \psi_k, \sum \xi_r \psi_r\right)$$

Because of the rectangular nature of the basic vector ψ_i, we find $(\phi,\phi) = \Sigma|\xi_k|^2$, and in the coordinate transformation with which we are concerned here, we find the square of the absolute value of the vector (the norm) to be invariable. This can be shown from the rectangular nature of Equation (27) using the expression in (29). According to Equation (31'), and because of the invariability of the norm,

$$\sum_k S_{ik}^* S_{jk} = \delta_{ij} \tag{34}$$

can be proven. If Equations (27) and (34) are expressed through the use of unit matrix E,

$$S'^*S = SS'^* = E \tag{35}$$

Generally the transformation satisfying such a relationship is termed a unitary transformation, and the matrix S which satisfies the relationship in Equation (35) is called a unitary matrix.

Let us consider the manner in which Equations (7) or (21), expressing intensity, are transformed by the above unitary transformation S. If vector ϕ is expressed by a new basic vector ψ_i as in Equation (28), we obtain

$$I = (\phi, A\phi) = (\sum \xi_i \psi_i, A \sum \xi_j \psi_j)$$

Using (25) and the orthogonal Equation (27), we readily find

$$I = \sum \lambda_i |\xi_i|^2 \tag{36}$$

Through (31), since ξ_i is expressed in (31) or (31')

$$\begin{aligned} I &= \sum \lambda_i |(\psi_i, \phi)|^2 \\ &= \sum_i \lambda_i |\sum_k S_{ki}^* u_k|^2 \end{aligned} \tag{37}$$

Showing the above results in matrix form, we have

$$\begin{aligned} I &= (\phi, A\phi) \\ &= (S\psi, AS\psi) = (\psi, S'^* AS\psi) \end{aligned}$$

but through (37), we find that S'*AS is a matrix where elements other than diagonal elements are zero. Letting D indicate this diagonal matrix,

$$S'^*AS = D \tag{38}$$

Here the diagonal elements of D are none other than the eigenvalues λ_1, λ_2, ..., λ_n.

An interesting result readily induced from (38) by (35) is

$$A = SDS^{*}$$ (39)

Or, for the matrix elements /439

$$a_{nm} = \sum \lambda_i S_{ni} S_{mi}^{*}$$ (40)

The most valuable of these results in considering the physical properties of intensity matrices are Equations (37) and (40). Let us consider these two findings.

(iii) <u>In terms of eigenvalue</u> λ_i <u>and eigenvector</u> $\psi_i(S_{1i}, S_{2i}, \ldots, S_{ni})$ <u>of a given intensity matrix A, the image intensity distribution is here</u>

$$I = \sum_i \lambda_i \left| \sum_k S_{ki}^{*} u_k \right|^2$$
$$u_k = \sin \frac{2\pi\alpha}{\lambda}\left(y - \frac{k\lambda}{2\alpha}\right) \Big/ \frac{2\pi\alpha}{\lambda}\left(y - \frac{k\lambda}{2\alpha}\right)$$ (37)

When the illumination is coherent, all eigenvalues save one are zero and

$$I = \lambda_0 \left| \sum S_k^{*} u_k \right|^2$$ (37')

Here, we find that $\sqrt{\lambda_0} S_k$ is a quantity corresponding to $F(k\lambda/2\alpha)$ in Equation (5). In partially coherent illumination, a number of terms appear in (27) but the major part is played obviously by the terms belonging to the maximum eigenvalue. In the completely incoherent case, each of the eigenvalues is of more or less the same importance. In § 4.2, in the special cases of uniform object brightness or when the brightness varies sinusoidally, we do not have a unitary transformation but an equation in the form of (37). When the object brightness is uniform, all the eigenvalues are equal.

As a result, depending on the values taken by the eigenvalues, the degree of illumination coherence can be estimated for the image formation. Formal though it may be, we should like to consider the following quantities. With λ_i as the eigenvalue, since the sum of all eigenvalues is equal to the trace of the intensity matrix and to the integral intensity I_0 of the image, the quantity

$$d = -\Sigma(\lambda_i/I_0)\log(\lambda_i/I_0) \tag{41}$$

with d = 0 in the case of coherent illumination, while in the case of incoherent illumination with uniform brightness, d is maximum. If N is the degree of freedom of the image,

$$d_0 = \log N$$

However, for N we find in the one-dimensional case $N = 2\alpha L/\lambda$ with image length L and $N = 4\alpha^2 S/\lambda^2$ with image area S in the two-dimensional case. Here, when the image area is very large, N increases indefinitely, but if we take d/d_0, we can deal with the upper limit for N. Then, when we take the quantity

$$\delta = (d_0 - d)/d_0$$

it changes from 1 to 0 as one proceeds from coherent to an incoherent uniform object image. This quantity is interesting in that it may be regarded as one measure of coherence. δ may be defined as the "degree of coherence".(7)

Moreover, according to von Neumann [8], $-\Sigma\lambda_n \log \lambda_n$ for matrix A is given by

$$-\Sigma\lambda_n \log \lambda_n = -\mathrm{Trace}\,(A \log A) \tag{43}$$

Here, the function log A of the matrix means the matrix obtained by substituting matrix A in each term of the series expanded logarithm. This furnishes a method of evaluating δ without a unitary transformation of matrix A.

---

(7) In a paper submitted to Kagaku [9], we discussed δ for images of objects with uniform transmissivity or brightness but according to (18'), one cannot distinguish between the semicoherent case when S > 1 and the incoherent case, and further study is required.

Now, let us summarize the results obtained from Equation (40):

(iv) <u>The arbitrary N^{th} degree positive Hermitian matrix, and consequently, intensity matrix element</u> a_{nm}, <u>are given by</u>

$$a_{nm} = \sum_i \lambda_i S_{ni} S_{mi}^* \qquad (40)$$

<u>from the positive real number</u> λ_i <u>of N and N orthogonal normal vectors</u> ψ_i $(S_{1i}, S_{2i}, \ldots, S_{Ni})$.

The number of variables designating an intensity matrix is determined by the above N eigenvalues and N eigenvectors. Since the vector component S_{ij} is generally complex, it may be thought of as $S_{ij} = r_{ij} \exp(\theta_{ij})$. Here, r_{ij}, θ_{ij} are real numbers. The number of variables designating these vector components is $2N^2$. Consequently, the number of variables determining a N^{th} degree intensity matrix is $(2N - 1)N$.

However, there is a conditional Equation (27) for N vectors to be positive orthogonal vectors. These conditional equation numbers are $N - 1/2N(N - 1) = 1/2N(N + 1)$, and thus:

(v) <u>The number R of independent variables to designate any arbitrary N^{th} degree intensity matrix is</u>:

$$R = N(2N+1) - \tfrac{1}{2}N(N+1) = \tfrac{1}{2}(3N+1) \qquad (44)$$

However, the simplest case of an intensity matrix is that of coherent illumination. This is because all eigenvalues but one λ_0 may be zero. In this case, only one vector is needed to give the matrix elements, and the number R(coherent) of independent variables is

$$R\,(\text{coherent}) = 2N \qquad (45)$$

These numbers of independent variables are fundamental qualities relating to entire intensity matrices determined by illumination coherence. However, we will show that all of these independent variables do not have any significance with respect to actual intensity matrices. Rather, intensity distribution itself is determined by 2N sampling values (the number of sampling values for an amplitude with coherent illumination is N). If we say that these 2N sampling values may take on any arbitrary values at all, the situation would be simple. In actuality they must be sampling values of the intensity distribution in the form of Equation (7) which can be formed from the above intensity matrices. This means the introduction of a correlation between the 2N sampling values. Consequently, the above intensity matrix properties are indispensable in inducing the amount of information concerning intensity distribution. Actual treatment will be reserved for discussion at a later opportunity.

(vi) Intensity distribution I(y) [Equation (7)] is determined by the sampling values at intervals $\lambda/4\alpha$ of intensity distribution, regardless of illumination coherence, and sample values and intensity matrices are related in the following manner:

$$I(y)=\sum_k I\left(\frac{k\lambda}{4\alpha}\right)\frac{\sin\frac{4\pi\alpha}{\lambda}\left(y-\frac{k\lambda}{4\alpha}\right)}{\frac{4\pi\alpha}{\lambda}\left(y-\frac{k\lambda}{4\alpha}\right)} \tag{46}$$

Here,

$$I\left(\frac{k\lambda}{4\alpha}\right)=\sum_n\sum_m a_{nm}\, u_n\left(\frac{k\lambda}{4\alpha}\right)u_m\left(\frac{k\lambda}{4\alpha}\right)$$

$$\begin{aligned}
k=2p:&\quad u_n(2p\lambda/4\alpha)=0,\ p\neq n\\
&\quad u_n(2p\lambda/4\alpha)=1,\ p=n\\
k=2p+1:&\quad u_n((2p+1)\lambda/4\alpha)\\
&\quad =(-1)^{p-n}/\{2(p-n)+1\}
\end{aligned}$$

To prove this, one finds the Fourier transformation of the intensity I(y) for (7) in § 5. The Fourier transformation of the right-side integral $u_n(y)\, u_m(y)$ has twice the range of the Fourier transformation of $u_n(y)$ alone, as follows, outside of which it is zero. The following integral is found by the method shown in the appendix

$$
\begin{aligned}
&I_{nm}(\omega)=\int_{-\infty}^{+\infty}\frac{\sin(\xi-n\pi)}{(\xi-n\pi)}\frac{\sin(\xi-m\pi)}{(\xi-m\pi)}e^{-i\omega\xi}d\xi\\
&\omega>2,\ \text{or}\ \omega<-2\ \text{and}\ I_{nm}\equiv 0\\
&0<\omega<2\\
&\quad I_{nm}=\frac{1}{2i}\frac{e^{-i\pi[n\omega-(n-m)]}-e^{-i\pi[m\omega+n-m]}}{n-m}\\
&0>\omega>-2\\
&\quad I_{nm}=\frac{1}{2i}\frac{e^{-i\pi[m\omega-(n-m)]}-e^{-i\pi[n\omega+n-m]}}{n-m}\\
&n=m\quad \text{When,}\ |\omega|>2\ \text{and}\ I_{nn}\equiv 0,\\
&0<\omega<2\quad I_{nn}=\pi\left(1-\frac{\omega}{2}\right)e^{-i\pi\omega n}\\
&0>\omega>-2\quad I_{nn}=\pi\left(1+\frac{\omega}{2}\right)e^{-i\pi\omega n}
\end{aligned}
\tag{47}
$$

Thus, we find that the range of $I(\omega)$ is limited to $4\alpha/\lambda$, and by using the Fourier series transformation in accordance with the sampling theorem in § 2, we can derive Equation (46).

Rewriting for the sampling values in (46), and letting B_k designate the k^{th} sampling value,

$$
\begin{aligned}
&B_{2p}=a_{pp},\quad B^{2p+1}=\\
&\sum_k\sum_r\frac{4}{\pi^2}a_{kr}(-1)^{k+r}/\{2(p-k)+1\}\{2(p-r)+1\}
\end{aligned}
\tag{48}
$$

And the relationship

$$
\frac{2\alpha}{\lambda}I_0=\sum_p B_{2p}=\sum_p B_{2p+1}=\sum_k a_{kk}
\tag{49}
$$

holds for the integral intensity I_0 for the entire image, from Equation (23).

6. CONCLUSION

These results supplemented a speech at the April 6, 1956 Applied Physics Joint Lecture Meeting symposium on "Contributions of Information Theory to Physics" [9]. The physical significance of "intensity matrices" presented above was explained. This work is more in the nature of a "prelude" to applications of information theory, and if it is to be used for actual calculation of amounts of information, intensity matrices must be expanded to cover two-dimensional cases. There remains the fundamental problem of changes in intensity matrices due to the phase differential method and the problem of intensity matrices in the case of optical systems containing aberrations. We intend to report on this ~~at~~ another time.

Moreover, the discussion above dealt with finite dimensions, and for completeness needs to take up infinite dimensions. For this, one must apply Hilbert spaces, but that was beyond the scope of this short paper.

I should like to thank Professor H. Takahashi of Tokyo University's Department of Physics and Assistant Professor K. Miyake of Tokyo University of Education for their consultation in compiling this paper and Professor H. Kubota of Tokyo University's Production Technology Institute for providing reference materials on the applications of information theory to optics.

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Note in Revision

It has come to our attention that D. Gabor in England independently describes the general application of Hermitian matrices to images. Information Theory, Third London Symposium, edited by Colin Cherry (1956, Butterworths Scientific Publications) Vol. 4. Optical Transmission by D. Gabor, pp. 26-33.

His method of derivation is different from that used in this paper, but the content is essentially the same and it is recommended that it be read in conjunction with this. In this paper, the author stated that matrix manipulation for intensity matrices in general cases was rather difficult, but later came to the conclusion that intensity matrices could be manipulated in the case of different objects or systems with aberrations through matrix transformation. This was presented at a lecture meeting of the Physics Department at Tokyo University on October 3 and the details are to be published. No discussions with Gabor took place, but the concepts are very similar.

Appendix 1

<u>Proof for Equation (8)</u>

Letting E(x) be the complex transmissivity of the object, A(x-z), the complex amplitude of the incident wave, according to Equation (1), the Fourier transformation of E(x)A(x-z) is

$$f(X,z)=\int_{-\infty}^{+\infty}E(x)A(x-z)\,e^{-2\pi iXx}\,dx \tag{1}$$

Also, from Equation (4) the amplitude distribution F(y) in the image plane is

$$F(y,z)=\int_{-\frac{\alpha}{\lambda}}^{+\frac{\alpha}{\lambda}}f(x,z)\,e^{2\pi iXy}\,dX \tag{2}$$

Substituting (1) in (2) and changing the integration sequence,

$$F(y,z)=\int_{-\infty}^{+\infty}E(x)A(x-z)\frac{\sin\frac{2\pi\alpha}{\lambda}(y-x)}{\pi(y-x)}\,dx \tag{3}$$

Here, if we let $u(y-x)=\sin\frac{2\pi\alpha}{\lambda}(y-x)/\pi(y-x)$, then

$$F(y,z)=\int_{-\infty}^{+\infty}E(x)A(x-z)u(y-x)\,dx$$

And if we next insert $F\left(\frac{n\lambda}{2\alpha},z\right)$, $F\left(\frac{m\lambda}{2\alpha},z\right)$ for points $y=\frac{n\lambda}{2\alpha}, \frac{m\lambda}{2\alpha}$ in Equation (7), we obtain Equation (8).

Appendix 2

(1) Incoherent case

Substituting the intensity matrix given by (15) in (7),

$$I(y)=\frac{2\alpha}{\lambda}A\sum_{n=-\infty}^{+\infty}\left\{\frac{\sin\frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right)}{\frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right)}\right\}^2.$$

Also, from (14')

$$I(y)=A\int_{-\infty}^{+\infty}\left\{\frac{\sin\frac{2\pi\alpha}{\lambda}(y-x)}{\pi(y-x)}\right\}^2dx$$
$$=\frac{2\alpha}{\lambda}A\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2(\eta-\xi)}{(\eta-\xi)^2}d\xi=\frac{2\alpha}{\lambda}A$$

Comparing both of these, we find that generally $\sum_{n=-\infty}^{+\infty}\frac{\sin^2(\eta-n\pi)}{(\eta-n\pi)^2}=1$ must be true. This relationship can also be derived from the series expansion of $1/(\sin^2\eta)$ (8)

Similarly, when the object brightness varies periodically, if we substitute (16) in (7),

$$I(\xi)=\frac{A\alpha}{\lambda}\left[1+(1-p)\sum_{n=-\infty}^{+\infty}\cos 2np\pi\frac{\sin^2(\xi-n\pi)}{(\xi-n\pi)^2}\right.$$
$$+\frac{1}{\pi}\sum_n{}'\sum_m{}'\frac{\sin(n-m)(1-p)\pi}{(n-m)}\cos(n+m)p\pi$$
$$\left.\times\frac{\sin(\xi-n\pi)\sin(\xi-m\pi)}{(\xi-n\pi)(\xi-m\pi)}\right]$$

Also, finding I(ξ) from Equation (14')

$$I(\xi)=\frac{A\alpha}{\lambda}\left[1+(1-p)\cos 2p\xi\right]$$

Setting these equal to each other, we obtain a single equality.

(8) For example, Formeln und Sätze für die speziellen Functionen der mathematischen Physik; (Formulas and Theorems for Special Functions of Mathematical Physics) (Springer 1948) p. 215.

(2) Partially Coherent Illumination

As to the image intensity distribution for the case where object transmissivity is constant, substituting a_{nm} in (7) according to (18') and (18"),

$$S>1,\quad I=\frac{K^2}{S}\left\{\sum_n \frac{\sin^2(\eta-n\pi)}{(\eta-n\pi)^2}\right\}=\frac{K^2}{S}$$

$$S<1$$

$$I=K^2\sum_n\sum_m \frac{\sin S(n-m)\pi}{S(n-m)\pi}\,\frac{\sin(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}$$

Also, for the equation corresponding to (14') when illumination is incoherent, we find

$$I(y)=\int\int \Gamma(x_1-x_2)E(x_1)E^*(x_2)A(x_1)A^*(x_2)\times u(x_1-y)u^*(x_2-y)\,dx_1dx_2$$

but, letting $E(x)A(x) = K$ and substituting $u(x - y)$ from (8') and $\Gamma(x_1 - x_2)$ from (17), we find, similar to the integral in (18),

$$S>1,\quad I=\frac{K^2}{S}:\ S<1,\quad I=K^2$$

Thus, we derive /442

$$\sum_n\sum_m \frac{\sin S(n-m)\pi}{S(n-m)\pi}\,\frac{\sin(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}=1$$

Appendix 3

$$1)\int_{-\infty}^{+\infty}\frac{\sin\frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right)}{\frac{2\pi\alpha}{\lambda}\left(y-\frac{n\lambda}{2\alpha}\right)}\frac{\sin\frac{2\pi\alpha}{\lambda}\left(y-\frac{m\lambda}{2\alpha}\right)}{\frac{2\pi\alpha}{\lambda}\left(y-\frac{m\lambda}{2\alpha}\right)}dy$$
$$=\frac{\lambda}{2\alpha}\delta_{nm}$$

Letting the variable equal $\frac{2\pi\alpha}{\lambda}=\xi$

$$\frac{\lambda}{2\pi\alpha}\int_{-\infty}^{+\infty}\frac{\sin(\xi-n\pi)}{(\xi-n\pi)}\frac{\sin(\xi-m\pi)}{(\xi-m\pi)}d\xi$$

Since the integrated function can be expressed as

$$=\frac{1}{(2i)^2}\frac{\{e^{i(\xi-n\pi)}-e^{-i(\xi-n\pi)}\}\{e^{i(\xi-m\pi)}-e^{-i(\xi-m\pi)}\}}{(\xi-n\pi)(\xi-m\pi)}$$

the original integral is the sum or difference of the following four integral forms

$$\int_{-\infty}^{+\infty}\frac{e^{iax}\,dx}{(x-\lambda_1)(x-\lambda_2)}=\pi i\frac{e^{ia\lambda_1}-e^{-ia\lambda_2}}{\lambda_1-\lambda_2}\quad(a>0)$$
$$(-)\pi i\frac{e^{ia\lambda_1}-e^{-ia\lambda_2}}{\lambda_1-\lambda_2}\quad(a<0)$$

To find this integral, one must select a method of integration such that exp(iax) vanishes, at infinity, and adopt Cauchy's principal value, in the case where the integral has an extremity on the real axis. Designating this kind of residue along the real axis by R_0, we must note the contribution of $\Sigma R+\frac{1}{2}$ to the value of the complex integral. However, R is the residue of the extremity inside the integration path. (Whittaker: Modern Analysis, page 117, 1935). The originally described results of integration were obtained by this means.

(2) That which is included in the integral in (16)

$$I_{nm}=\int_{-\infty}^{+\infty}\cos 2p\xi\,\frac{\sin(\xi-n\pi)}{\xi-n\pi}\,\frac{\sin(\xi-m\pi)}{\xi-m\pi}\,d\xi$$

As above, this is separated into complex integrals containing designated functions. The results are immediately reached by applying the above formula to each, or

$$p>1 \qquad I_{nm}=0$$

$$0 \leqq p \leqq 1 \qquad n \neq m$$

$$I_{nm}=\frac{\sin\pi\,(n-m)(1-p)}{n-m}\cos\pi(n+m)p$$

$$n=m \qquad I_{nm}=\pi(1-p)\cos 2\pi np$$

(3) Integral included in (18)

$$I=\int_{-\infty}^{+\infty}\frac{\sin S(\xi-\eta)}{S(\xi-\eta)}\,\frac{\sin(\xi-n\pi)}{\xi-n\pi}\,d\xi$$

$$S>1 \qquad I=\frac{\pi}{S}\,\frac{\sin(\eta-n\pi)}{\eta-n\pi}$$

$$S<1 \qquad I=\frac{\pi}{S}\,\frac{\sin(\eta-n\pi)}{\eta-n\pi}$$

(4) Definite integral required for calculation of (18)

$$I=\int_{-\infty}^{+\infty}\cos p\xi\,\frac{\sin S(\xi-\eta)}{S(\xi-\eta)}\,\frac{\sin(\xi-n\pi)}{\xi-n\pi}\,d\xi$$

(i) $S \geqq 1,\ p>(1+S);\ I\equiv 0$

(ii) $S>1 \quad (1+S)>p>(S-1)$, および, $S<1$

$(1+S)>p>(1-S)$

$$I=\frac{\pi}{2S}\,\frac{\sin[S(\eta-n\pi)+pn\pi]+\sin[\eta-n\pi-p\eta]}{\eta-n\pi}$$

(iii) $S>1 \quad (S-1)>p>0$:

$$I=\frac{\pi}{S}\,\frac{\sin(\eta-n\pi)}{\eta-n\pi}\cos p\eta$$

(iv) $S<1.\ (1-S)>p>0$:

$$I=\frac{\pi}{S}\,\frac{\sin S(\eta-n\pi)}{\eta-n\pi}\cos pn\pi.$$

(19) is reduced to a single integral using the results above, i.e.:

(i) $S>1,\ p>(1+S)$: $\quad a_{nm}\equiv 0$

(ii) $S>1,\ (1+S)>p>(S-1)$

$S<1,\ (1+S)>p>(1-S)$

$$a_{nm}=\frac{K^2}{2\pi S}\int_{-\infty}^{+\infty}\cos p\eta$$

$$\times\frac{\{\sin[S(\eta-n\pi)+pn\pi]+\sin[\eta-n\pi-p\eta]\}}{\eta-n\pi}$$

$$\times\frac{\sin(d\eta-m\pi)}{\eta-m\pi}\,d\eta$$

(iii) $S>1 \quad (S-1)>p>0$

$$a_{nm}=\frac{K^2}{\pi S}\int_{-\infty}^{+\infty}\cos^2 p\eta\,\frac{\sin(\eta-\eta\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\eta)}{\eta-m\pi}\,d\eta$$

(iv) $S<1 \quad (1-S)>p>0$

$$a_{nm}=\frac{K^2}{\pi S}\cos pn\pi\int_{-\infty}^{+\infty}\cos p\eta\,\frac{\sin S(\eta-n\pi)}{\eta-n\pi}\times\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\eta$$

Of these integrals (iii) and (iv) are readily found from Formulas (2) and (4) as given above. The integral in (ii) is transformed into the following:

$$\frac{2\pi S}{K^2}a_{nm}=\sin pn\pi\int_{-\infty}^{+\infty}\cos p\eta\,\frac{\cos S(\eta-n\pi)}{\eta-n\pi}\times\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\eta \qquad \text{(a)}$$

$$+\cos pn\pi\int_{-\infty}^{+\infty}\cos p\eta\,\frac{\sin S(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\eta \qquad \text{(b)}$$

$$+\int_{-\infty}^{+\infty}\cos^2 p\eta\,\frac{\sin(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\eta \qquad \text{(c)}$$

$$-\int_{-\infty}^{\infty}\sin p\eta\cos p\eta\,\frac{\cos(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,dy. \qquad \text{(d)}$$

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In integrals (b) and (c), the above formulas can be used, but new calculations are necessary for (a) and (d).

The values of these four definite integrals are then:

$$\int_{-\infty}^{+\infty}\cos p\eta\,\frac{\sin S(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\eta$$

$$n\neq m \quad =\frac{\sin(n-m)(S+1-p)\frac{\pi}{2}}{n-m}\cos[(n+m)p+(S-1)(n-m)]\frac{\pi}{2}$$

$$n=m \quad =\frac{\pi}{2}(S+1-p)\cos np\pi$$

$$\int_{-\infty}^{+\infty}\cos p\eta\,\frac{\cos S(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\eta$$

$$n\neq m \quad =(-)\frac{\sin(n-m)(p+1-S)\frac{2}{\pi}}{n-m}\times\sin[(n+m)p+(S+1)(n-m)]\frac{\pi}{2}$$

$$n=m \quad =(-)\frac{\pi}{2}(p+1-S,\ \sin np\pi$$

$$\int_{-\infty}^{+\infty}\sin 2p\eta\,\frac{\cos(\eta-n\pi)}{\eta-n\pi}\,\frac{\sin(\eta-m\pi)}{\eta-m\pi}\,d\eta$$

$$(1+S)>p>1\begin{cases}n\neq m & =0\\ n=m & =\pi\cos 2np\pi\end{cases}$$

$1>p>(S-1)$ 又は $(1-S)$

$$\begin{cases}n\neq m & =\dfrac{\sin(m-m)p\pi}{n-m}\cos[(n+m)p+n-m]\pi\\ n=m & =\pi p\cos 2np\pi\end{cases}$$

continued

continued

$$\int_{-\infty}^{+\infty} \cos 2p\eta \, \frac{\sin(\eta - m\pi)}{\eta - n\pi} \, \frac{\sin(\eta - m\pi)}{\eta - m\pi} \, d\eta$$

$(1+S) > p > 1 \quad n \neq m, \quad n = m, \quad 1 = 0$

$1 > p > (S-1)$ 又は $(1-S)$

$$\begin{cases} n \neq m & = \dfrac{\sin(n-m)(1-p)\pi}{n-m} \cos(n+m)p\pi \\ n = m & = \pi(1-p)\cos 2np\pi \end{cases}$$

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